On the Jones polynomials of checkerboard colorable virtual knots

Naoko Kamada
Department of Mathematics and Statistics
University of South Alabama

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Abstract

In this paper we study the Jones polynomials of virtual links and abstract links. It is proved that a certain property of the Jones polynomials of classical links is valid for virtual links which admit checkerboard colorings.

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1 Introduction

In 1996, L. H. Kauffman introduced the notion of a virtual knot, which is motivated by study of knots in a thickened surface and abstract Gauss codes, cf. [8, 9]. M. Goussarov, M. Polyak, and O. Viro [1] proved that the natural map from the category of classical knots to the category of virtual knots is injective; namely, if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots. Thus, virtual knot theory is a generalization of knot theory. It is also found in their paper [1] that the notion of a virtual knot is helpful to study of finite type invariants.

Kauffman defined the Jones polynomial of a virtual knot, which is also called the normalized bracket polynomial or the f-polynomial (cf. [9]). In this paper, according to [9], we call it the f-polynomial instead of the Jones polynomial, since the definition is different from Jones' in [2, 3]. Finite type invariants derived from the f-polynomials are studied in [9], and it is proved that a certain property of them (Corollary 14 of [9]) is hold in the category of virtual knots.

The f-polynomial (Jones polynomial) of a virtual link is quite different from f-polynomials of classical links. For a Laurent polynomial f on valuable A, we denote by $\mathrm{EXP}(f)$ the set of integers appearing as exponents of f. For example, if $f = 3A^{-2} + 6A - 7A^5$, then $\mathrm{EXP}(f) = \{-2, 1, 5\}$. It is well-known

that for a classical link L with n components, the f-polynomial satisfies that $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ if n is odd, and $\mathrm{EXP}(f) \subset 4\mathbf{Z}+2$ if n is even. However, this is not true for a virtual knot/link in general. In this paper we introduce the notion of *checkerboard coloring* of a virtual link diagram as a generalization of checkerboard coloring of a classical link diagram.

Theorem 1 *Let f be the f-polynomial of a virtual link L with n components. Suppose that L has a virtual link diagram which admits a checkerboard coloring. Then* $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ *if n is odd, and* $\mathrm{EXP}(f) \subset 4\mathbf{Z}+2$ *if n is even.*

For example the virtual knot diagram illustrated in Figure 1 (a) admits a checkerboard coloring and the f-polynomial is $A^4 + A^{12} - A^{16}$. So $\mathrm{EXP}(f) \subset 4\mathbf{Z}$. On the other hand, virtual knot diagram illustrated in Figure 1 (b) does not admit a checkerboard coloring and the f-polynomial is $-A^{10} + A^6 + A^4$. Theorem 1 implies that this diagram is never equivalent to a diagram that admits a checkerboard coloring.

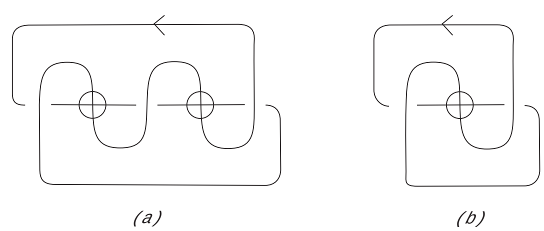

Figure 1:

If a virtual link diagram is alternating (the definition is given later), then the diagram admits a checkerboard coloring. Thus we have the following.

Corollary 2 *Let f be the f-polynomial of a virtual link L with n components. Suppose that L has an alternating virtual link diagram. Then* $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ *if n is odd, and* $\mathrm{EXP}(f) \subset 4\mathbf{Z}+2$ *if n is even.*

By this corollary, we see that the virtual knot represented by Figure 1 (b) is not equivalent to an alternating diagram.

2 Virtual link diagram and abstract link diagram

A *virtual link diagram* is a closed oriented 1-manifold generically immersed in $\mathbf{R}^2$ such that each double point has information of a crossing (as in classical

knot theory) or a virtual crossing which is indicated by a small circle around the double point. The moves of virtual link diagrams illustrated in Figure 2 are called *generalized Reidemeister moves*. Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We call the equivalence class of a virtual link diagram *a virtual link*.

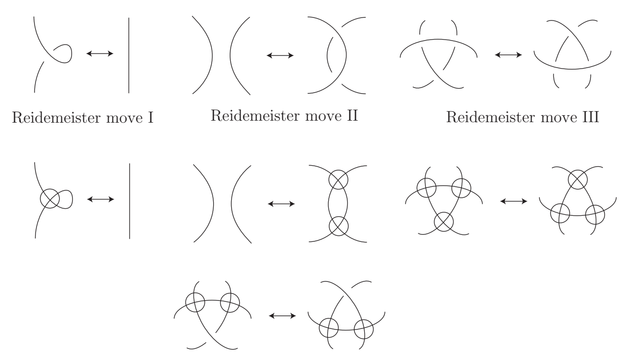

Figure 2:

A pair $P = (\Sigma, D)$ of a compact oriented surface Σ and a link diagram D in Σ is called an *abstract link diagram* (ALD) if $|D|$ is a deformation retract of Σ, where $|D|$ is a graph obtained from D by replacing each real/virtual crossing point with a vertex. For an ALD, $P = (\Sigma, D)$, if there is an orientation preserving embedding $f : \Sigma \to F$ into a closed oriented surface F, $f(D)$ is a link diagram in F. We call it a *link diagram realization* of P in F. In Figure 3, we show two abstract link diagrams and their link diagram realizations. Two ALDs $P = (\Sigma, D)$, $P' = (\Sigma', D')$ are related by an *abstract Reidemeister move* (of type I, II or III) if there is a closed oriented surface F and link diagram realizations of P and P' in F which are related by a Reidemeister move (of type I, II or III) in F. Two ALDs are *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call the equivalence class of an ALD *an abstract link*.

In [6] a map

$$\phi : \{\text{virtual link diagrams}\} \longrightarrow \{\text{ALDs}\}$$

was defined. The idea of this map is illustrated in Figure 4. Refer to [6] for the definition. We call $\phi(D)$ an *ALD associated with* a virtual link diagram D. The ALDs in Figure 3 (a) and (b) are ALDs associated with the virtual link diagrams in Figure 1 (a) and (b) respectively.

Theorem 3 *([6]) The map ϕ induces a bijection*

$$\Phi : \{\textit{virtual links}\} \longrightarrow \{\textit{abstract links}\}$$

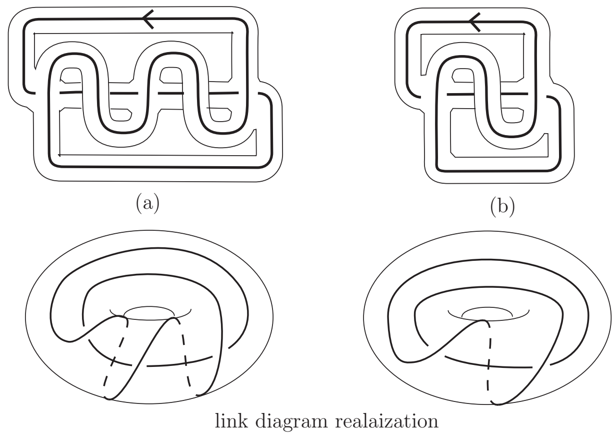

Figure 3:

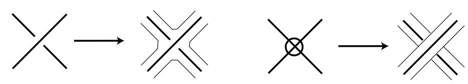

Figure 4:

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Let $P = (\Sigma, D)$ be a pair of a compact oriented surface Σ and a link diagram D in Σ. A *checkerboard coloring* is a coloring of the all components of $\Sigma - |D|$ by two colors, say black and white, such that two components of $\Sigma - |D|$ which are adjacent by an edge of D have always distinct colors.

We say that a virtual link diagram *admits a checkerboard coloring* or it is *checkerboard colorable* if the associated ALD admits a checkerboard coloring.

3 The f-polynomials of abstract link diagrams

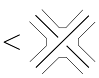
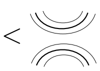
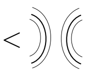
An ALD, $P = (\Sigma, D)$, is said to be *unoriented* if the diagram D is unoriented. There is a unique map

$$< \quad >: \{\text{unoriented ALDs}\} \longrightarrow \Lambda = \mathbf{Z}[A, A^{-1}]$$

satisfying the following rules.

(i) $< T >_F = 1$ where T is a one-component trivial ALD,

(ii) $< T \amalg D >= (-A^2 - A^{-2}) < D >$ if D is not empty, where $\amalg$ means the disjoint union, and

(iii) $<$  $>= A <$  $> + A^{-1} <$  $>.$

Then $< \ >$ is an invariant under abstract Reidemeister moves II and III. We call it the *Kauffman bracket polynomial* of ALD, cf. [4].

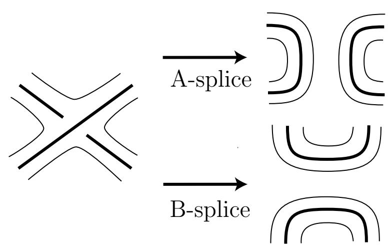


Figure 5:

Let $P = (\Sigma, D)$ be an unoriented ALD. Replacing the neighborhood of a double point as in Figure 5, we have another unoriented ALD. We call it an unoriented ALD obtained from D by doing an *A-splice* or *B-splice* at the crossing point. An unoriented trivial ALD obtained from P by doing an A-splice or B-splice at each crossing point is said to be a *state* of P. From the definition of $< \ >$, we see

$$< P >= \sum_S A^{\natural(S)} (-A^2 - A^{-2})^{\sharp(S)-1},$$

where S runs over all of states of D, $\natural(S)$ is the number of A-splice minus that of B-splice used for obtaining S and $\sharp(S)$ is the number of components of S.

For an ALD, $P = (\Sigma, D)$, the writhe $\omega(P)$ is defined by the number of positive crossings minus the number of negative crossings. Then we define the *normalized bracket polynomial* or the *f-polynomial* of P by

$$f_P(A) = (-A^3)^{-\omega(P)} < P > .$$

By normalizing by $(-A^3)^{-\omega(P)}$, this value is preserved under abstract Reidemeister moves of type I. Thus this is an invariant of an abstract link. This invariant was defined in [4], where it is called the Jones polynomial of P. It should be noted that the bijection Φ preserves the f-polynomial.

4 Proof of Theorem 1

Let p be a crossing point of an ALD, $P = (\Sigma, D)$. Let $P_0 = (\Sigma_0, D_0)$ and $P_\infty = (\Sigma_\infty, D_\infty)$ be ALDs obtained from P by splicing at p orientation coherently and orientation incoherently, respectively. Note that D_∞ does not inherit an orientation from D. The crossing point p is either (i) a self-intersection of an immersed loop of D or (ii) an intersection of two immersed loops. Let α and α' be the immersed open arcs obtained from the loop (in case (i)) or from the two loops (in case (ii)) by cutting at p. Choose one of them, say α, and we give an orientation to D_∞ which is induced from that of D except α (and hence the orientation is reversed on α). Let C be the set of crossing points of D, except p, such that the sign of the crossing point does not change in D and D_∞; in other word, at each crossing point belonging to C, both of the two intersecting arcs are contained in $D - \alpha$ or both of them are in α. Let C' be the set of crossing points of D, except p, such that the sign of the crossing point changes in D and D_∞; in other word, at each crossing point belonging to C', one of the two intersecting arcs is contained in $D - \alpha$ and the other is in α. Let k (or ℓ, resp.) be the number of positive crossings of C (resp. C') minus the number of negative crossings of C (resp. C').

Lemma 4 *In the above situation, let f, f_0 and f_∞ be the f-polynomials of P, P_0 and P_∞, respectively. Then we have*

$$f = \begin{cases} -A^{-2} f_0 - (-A^3)^{-2\ell} A^{-4} f_\infty, & \text{if } p \text{ is a positive crossing,} \\ -A^{+2} f_0 - (-A^3)^{-2\ell} A^{+4} f_\infty, & \text{if } p \text{ is a negative crossing.} \end{cases}$$

Proof. If p is a positive crossing, then the writhes are $\omega(D) = k + \ell + 1$, $\omega(D_0) = k + \ell$ and $\omega(D_\infty) = k - \ell$. Since $< P >= A < P_0 > +A^{-1} < P_\infty >$, we have the result. The case that p is a negative crossing is similar. □

Remark. In the remark of Section 5 of [9](page 677), an equation which is similar to Lemma 4 is given. However, it seems to be forgotten there to take account of the term $(-A^3)^{-2\ell}$. In consequence, the recursion formula of Theorem 13 of [9] is as follows:

$$v_n(G_*) = \sum_{k=0}^{n-1} \frac{2^{n-k}}{(n-k)!} \{(1-(-1)^{n-k}) v_k(G_0) + \{(2-3\ell)^{n-k} - (-2-3\ell)^{n-k}\} v_k(G_\infty)\}.$$

By this formula, Corollary 14 of [9] is still true.

Corollary 5 (cf. Theorem 13 of [9]) *Let f be the f-polynomial of an ALD with n components. Then $f(1) = (-2)^{n-1}$. In particular, f-polynomials of ALDs are not zero.*

Proof. It follows from Lemma 4 by induction on the number of (real) crossing points. □

Since Φ preserves the f-polynomials, Theorem 1 is equivalent to the following theorem.

Theorem 6 *Let f be the f-polynomial of an ALD, $P = (\Sigma, D)$, with n components. Suppose that P admits a checkerboard coloring. Then* $\mathrm{EXP}(f) \subset 4\mathbf{Z}$ *if n is odd, and* $\mathrm{EXP}(f) \subset 4\mathbf{Z} + 2$ *if n is even.*

Proof. For a state S of P, we define $I(S)$ by

$$I(S) = A^{\natural(S)}(-A^2 - A^{-2})^{\sharp(S)-1}$$

so that the bracket polynomial of P is the sum of $I(S)$ for all states S. Let $\mathrm{ind}(S)$ be a value in $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ such that $I(S) \subset 4\mathbf{Z} + \mathrm{ind}(S)$.

Every state of P has a unique checkerboard coloring induced from the checkerboard coloring of P, see Figure 6. (Figure 7 is an example of an ALD with a checkerboard coloring and a state with the induced checkerboard coloring.) Using this fact, we prove that $\mathrm{ind}(S) = \mathrm{ind}(S')$ for any states S and S' of P. It is sufficient to prove this in a special case that S and S' are the same state except a crossing point, say p, of D where S and S' are as in Figure 8. For this state S, there are two cases (A) and (B) as in Figure 9. The case (C) does not occur, because a state as in (C) does not have a checkerboard coloring induced from the checkerboard coloring of P. In both cases (A) and (B), we have $I(S') = A^{\natural(S)\pm 2}(-A^2 - A^{-2})^{\sharp(S)-1\pm 1}$ and $\mathrm{ind}(S) = \mathrm{ind}(S')$.

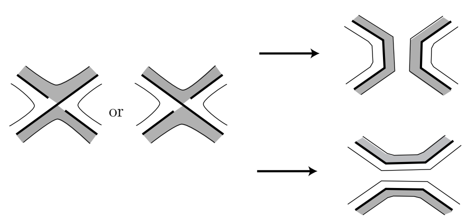


Figure 6:

Now we have that $\mathrm{EXP}(f) \subset 4\mathbf{Z} + i$ where $i = \mathrm{ind}(S)$ for any state S of P. We denote this number i by $\mathrm{ind}(f)$. The remaining task is to prove this index is 0 if n is odd, and 2 if n is even. This is proved by induction on the number of (real) crossing points of P. If P has no real crossing points, then this is obvious by the definition of the f-polynomial. If there is a crossing point, say p, apply Lemma 4. Note that P_0 and P_∞ have checkerboard colorings, and $\mathrm{EXP}(f_0) \subset 4\mathbf{Z} + \mathrm{ind}(f_0)$ and $\mathrm{EXP}(f_\infty) \subset 4\mathbf{Z} + \mathrm{ind}(f_\infty)$. Since $f \neq 0$ and $f_0 \neq 0$ (Corollary 5), it follows

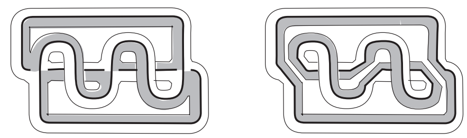


Figure 7:

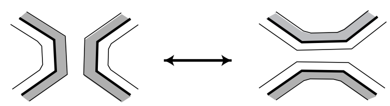


Figure 8:

from the equation in Lemma 4 that $\mathrm{ind}(f) = \mathrm{ind}(f_0) + 2 \in \mathbf{Z}_4$. The ALD P_0 has fewer crossing points than P and has a checkerboard coloring. By induction hypothesis, $\mathrm{ind}(f_0)$ is 0 if n' is odd, and 2 if n' is even, where n' is the number of components of P_0. Since $n' = n \pm 1$, we have that $\mathrm{ind}(f)$ is 0 if n is odd, and 2 if n is even. $\square$

5 Alternating virtual link diagrams and ALDs

An ALD or a virtual link diagram is *alternating* if we meet over and under crossing points alternatively when we travel along each component of the diagram twice.

Lemma 7 *For an ALD, $P = (\Sigma, D)$, the following conditions are equivalent.*

(i) *By applying crossing changes, P changes into an alternating ALD.*

(ii) *P has a checkerboard coloring.*

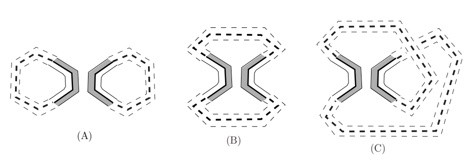


Figure 9:

Proof of Lemma 7. If P has a checkerboard coloring, change each real crossing according to the coloring as in the most left figure of Figure 6. Conversely if P is an alternating ALD, then give a checkerboard coloring near each crossing point as in the picture, which is extended to a checkerboard coloring of P. $\square$

Proof of Corollary 2. It follows from Theorem 1 and Lemma 7. $\square$

Remark. M. B. Thistlethwaite [11] and K. Murasugi [10] showed that the f-polynomial (Jones polynomial) of a non-split alternating link is alternating, namely it is in a form of $A^\alpha \sum c_i A^{4i}$ such that $c_i c_j \geq 0$ for $i \equiv j \pmod 2$ and $c_i c_j \leq 0$ for $i \not\equiv j \pmod 2$. This does not hold in virtual knot theory. The f-polynomial of a virtual knot in Figure 10 is $A^{12} + 3A^{16} - 4A^{20} + 3A^{24} - 4A^{28} + 4A^{32} - 3A^{36} + A^{40}$.

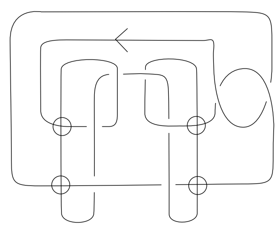


Figure 10:

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